Precept 10

1. Let $f(x) = x^2$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. Find the Lagrange interpolant $p_2(x)$ and the Barycentric interpolant $q_2(x)$ of f at the points x_0, x_1, x_2 (That is write x^2 in each of the corresonding forms.)

2. The Vandermonde matrix is defined as:

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

Prove that the Vandermonde matrix is invertible. Hint: relate V to an interpolant polynomial and use uniqueness of the interpolant polynomial.

3. Recall that Simpson's rule on $[0, 1]$ is

$$\int_0^1 f(x)\,dx \approx \frac{1}{6}\Big(f(0) + 4f\big(\tfrac{1}{2}\big) + f(1)\Big),$$

and that the Newton–Cotes error bound for nodes $x_0, \ldots, x_n$ on $[a, b]$ is

$$|E_n(f)| \leq \frac{\max\limits_{x\in[a,b]} |f^{(n+1)}(x)|}{(n+1)!} \int_a^b \left| \prod_{i=0}^{n} (x - x_i) \right| dx.$$

Using this bound with $n = 2$ and the Simpson nodes

$$x_0 = 0, \qquad x_1 = \frac{1}{2}, \qquad x_2 = 1,$$

show that

$$\Big| \int_0^1 f(x)dx - \frac{1}{6}\Big(f(0) + 4f\big(\tfrac{1}{2}\big) + f(1)\Big)\Big| \leq \frac{1}{192} \max_{x\in[0,1]} |f^{(3)}(x)|.$$

4. Given $n+1$ data pairs $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, define for $j = 0, 1, \ldots, n$ the quantities

$$\rho_j = \prod_{i\neq j} (x_j - x_i),$$

and let also

$$\psi(x) = \prod_{i=0}^{n} (x - x_i).$$

(a) Show that $\rho_j = \psi'(x_j)$.

(b) Show that the interpolating polynomial of degree at most n is given by

$$p_n(x) = \psi(x) \sum_{j=0}^{n} \frac{y_j}{(x - x_j)\psi'(x_j)}.$$