Precept 11

Q1. Orthogonal polynomials with weight $w(x) = -\ln x$ on $(0, 1)$

Consider the weighted inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, (-\ln x)\, dx.$$

(a) Show, using integration by parts, that for every integer $k \geq 0$,

$$\int_0^1 x^k (-\ln x)\, dx = \frac{1}{(k+1)^2}.$$

(b) Using Gram–Schmidt on the basis $\{1, x, x^2\}$ with respect to this inner product, construct orthogonal polynomials of degrees $0, 1, 2$ with positive leading coefficients.

Q2. Finite-difference endpoint correction

Recall that the Euler–Maclaurin formula states that

$$T(h) = \int_a^b f(x)\, dx + \frac{h^2}{12}\big(f'(b) - f'(a)\big) + \mathcal{O}(h^4),$$

where $T(h) = \sum_{i=0}^{n} f(x_i)h - \frac{h}{2}(f(x_0) + f(x_n))$, $x_i = a + ih$, $h = \frac{b-a}{n}$ is the result of the composite trapezoidal rule.

(a) Using a finite difference scheme to approximate $f'(a)$ and $f'(b)$, explain how you can create an integration scheme of order $\mathcal{O}(h^4)$ by evaluating f at most $n + 3$ times.

(b) Using only the $n+1$ samples of f used to compute the trapezoidal rule, explain how you can create an integration scheme of order $\mathcal{O}(h^4)$ (use an endpoint correction to the trapezoidal rule, not Richardson extrapolation).

Q3. Spectral differentiation

Recall that the Chebyshev polynomials are defined from the three-term recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x).$$

Spectral differentiation is a method to estimate the derivative of a smooth function f given samples of the function on a grid. It works by evaluating the derivative of the interpolant of the function.

(a) Find a recurrence formula for the derivative of Chebyshev polynomials. (it will involve both $T_j(x)$ and $T_j'(x)$)

(b) Let $p(x) = \sum_{j=0}^{n} c_j T_j(x)$ be a Chebyshev series of degree n. Describe an $\mathcal{O}(n)$ algorithm to evaluate $p'(x^*)$ at a given point x^*.

Q4. Let $x_0, \ldots, x_n \in [a, b]$ be distinct nodes. We defined the quadrature weights in two different ways:

(A) **Moment–matching weights.** The weights $w_0, \ldots, w_n$ are defined as the unique solution of

$$\sum_{k=0}^{n} w_k \, x_k^j = \int_a^b x^j \, dx, \qquad j = 0, 1, \ldots, n.$$

(B) **Interpolatory–polynomial weights.** Let $\ell_k(x)$ be the Lagrange basis polynomial at the nodes $x_0, \ldots, x_n$. Define the quadrature rule

$$Q(f) := \int_a^b p(x) \, dx, \qquad p(x) = \sum_{k=0}^{n} f(x_k) \, \ell_k(x),$$

and set

$$w_k := \int_a^b \ell_k(x) \, dx.$$

(a) Show that the weights obtained from (A) and (B) are identical.

(b) Conclude that the interpolatory quadrature rule with $n + 1$ nodes is exact for all polynomials of degree $\leq n$.

Q5. (Symmetric two–point quadrature on $[-1, 1]$).

A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x) \, dx \; \approx \; w_0 f(-\alpha) + w_1 f(\alpha).$$

The formula is required to be exact whenever f is a polynomial of degree 1.

(a) Show that $w_0 = w_1 = 1$, independent of the value of α.

(b) Show that there is one particular value of α for which the formula is exact also for all polynomials of degree 2. Find this α, and show that, for this value, the formula is also exact for all polynomials of degree 3.