Precept 12

Q1. Second-order ODE and Euler's method

Consider the second-order initial value problem

$$y'' + 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Rewrite this as a first-order system of ODEs.

(b) Find a matrix A such that Euler's method with step size h gives the iteration $u_{n+1} = Au_n$, where u_n is the approximation at time $t_n = nh$.

Q2. Matrix exponential and stability

Let $A \in \mathbb{R}^{n\times n}$, and consider the system of differential equations

$$\frac{dy}{dt} = -Ay, \qquad y(0) = y_0.$$

for $y(t) : \mathbb{R}^+ \to \mathbb{R}^n$.

(a) Show that the solution $y(t)$ is given by

$$y(t) = e^{-At}y_0,$$

where $e^{-At} = \sum_{k=0}^{\infty} \frac{(-At)^k}{k!}$ is the matrix exponential.

(b) Conclude that if A is symmetric positive-definite, then the solution $y(t)$ decays exponentially, that is $\|y(t)\|_2 \leq e^{-ct}\|y_0\|_2$ for some $c > 0$ (hint: Diagonalize $A = Q\Lambda Q^T$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and Q is orthogonal)

(c) Still assuming that A is symmetric positive-definite, suppose we apply Euler's method to this system with step size h. For what range of h is the method stable? (i.e., when does the numerical solution also decay exponentially?)

Q3. Non-uniqueness and Picard's theorem

Suppose that m is a fixed positive integer. Consider the initial value problem

$$y' = y^{2m/(2m+1)}, \qquad y(0) = 0.$$

(a) Show that this problem has infinitely many continuously differentiable solutions $y : [0, \infty) \to \mathbb{R}$. Hint: For any $a \geq 0$, consider

$$y_a(x) := \begin{cases} 0, & 0 \leq x \leq a, \\ \left(\dfrac{x-a}{2m+1}\right)^{2m+1}, & x \geq a. \end{cases}$$

(b) Explain why this does not contradict Picard's theorem.

Q4. Consider the trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}\big(f_{n+1} + f_n\big), \qquad f_n := f(x_n, y_n), \quad h = x_{n+1} - x_n,$$

for the numerical solution of $y' = f(x, y)$ with $y(0) = y_0$.

Define the truncation error T_n by

$$y(x_{n+1}) = y(x_n) + \frac{h}{2}\big(f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\big) + hT_n,$$

where y is the exact solution. By integrating by parts, one can show that

$$T_n = -\frac{1}{12}h^2 y'''(\xi_n)$$

for some $\xi_n \in (x_n, x_{n+1})$ (you can use this part without proof).

(a) Suppose that f satisfies the Lipschitz condition

$$|f(x, u) - f(x, v)| \le L|u - v|$$

for all real x, u, v, and that $|y'''(x)| \le M$ for some constant $M > 0$. Show that the global error $e_n := y(x_n) - y_n$ satisfies

$$|e_{n+1}| \le |e_n| + \frac{1}{2}hL\big(|e_{n+1}| + |e_n|\big) + \frac{1}{12}h^3 M.$$

(b) Assume a constant step size $h > 0$ with $hL < 2$ and $y_0 = y(x_0)$. Show that

$$|e_n| \le \frac{h^2 M}{12L}\left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\right)^n - 1\right].$$