Precept 5: Midterm practice problems

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1 Definitions and Concepts

Make sure you can define and explain all these definitions and concepts.

Definitions:

- LU, partially pivoted LU
- reduced SVD, full SVD, truncated SVD
- thin(/reduced/economy) QR, full QR
- Cholesky decomposition
- Operator norms: $\|\cdot\|_p$ for $p = 1, 2, \infty$ (both definition and how to compute them)
- Eckart-Young-Mirsky theorem
- Gauss transformations, Givens Rotations, Householder reflections

Concepts:

- What is the complexity of LU for an $n \times n$ matrix? QR for an $m \times n$ matrix? matrix-vector multiply for an $m \times n$ matrix?
- Suppose we run bisection on a function f and an interval $[a, b]$. Under what condition does the Bisection method converge? At what rate?
- Under what condition does Newton's method converge quadratically locally? What about linearly? (Can you prove this?)
- Under what condition does a fixed point iteration $x_{k+1} = f(x_k)$ converge linearly locally? What about quadratically? (Can you prove this?)

2 Iterative solution (5 pts)

Let $A, B \in \mathbb{R}^{n \times n}$ be invertible, and B is close to A in the sense that $\sigma_1(B^{-1}A - I) = \rho < 1$. That is, ρ is the largest singular value of $C := (B^{-1}A - I)$, and ρ is less than one. Let $b \in \mathbb{R}^n$ and let $c = B^{-1}b$. Consider the iteration:

$$x_{k+1} = x_k - B^{-1}Ax_k + c$$

1. (2pts) Find a fixed point x^* of the iteration.
2. (3pts) Define the error $e_k = \|x_k - x^*\|_2$. For any x_0, show that $e_k \leq C\rho^k$ for some C and ρ (find these values).

3 Transformations

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix}$$

with $x_k \neq 0$. Find $u, v \in \mathbb{R}^n$ such that:

$$(I - uv^T)x = y$$

and $(I - uv^T)$ is upper triangular.

4 Least Squares with Block Elimination

Let

$$A \in \mathbb{R}^{n\times p}, \qquad B \in \mathbb{R}^{n\times q}, \qquad C \in \mathbb{R}^{m\times q}, \qquad x \in \mathbb{R}^p,\ y \in \mathbb{R}^q,\ d \in \mathbb{R}^m.$$

Assume that *each of A, B, C has full column rank.* Consider the coupled least-squares objective

$$\min_{x\in\mathbb{R}^p,\, y\in\mathbb{R}^q} \|Ax + By\|_2^2 \ +\ \|Cy - d\|_2^2.$$

1. Write down the least squares problem in standard form. That is find $\hat{A}, \hat{b}$ such that the least squares problem is equivalent to $\min_{\hat{x}} \|\hat{A}\hat{x} - \hat{b}\|_2^2$.
2. Find the (block) normal equations.
3. Do a step of block Gaussian elimination to find a linear system that y satisfies. That is, find H and v such that $Hy = v$.
4. Recover x from y.

5 First Row of Matrix Inverse

Suppose $A \in \mathbb{R}^{n\times n}$ is invertible, and we have its LU $A = LU$. Describe an algorithm to compute the first row of A^{-1} in $\mathcal{O}(n^2)$

6 Leading Principal Submatrices and Conditioning (SPD case)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD). Denote its eigenvalues by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Let $A_{11} \in \mathbb{R}^{k\times k}$ be the leading principal submatrix of A (first k rows and columns), with eigenvalues

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k.$$

Cauchy's interlacing theorem: For a symmetric matrix A and its leading principal submatrix A_{11},

$$\lambda_j \leq \mu_j \leq \lambda_{n-k+j}, \qquad j = 1, \dots, k.$$

1. Show that for SPD A, the singular values of A equal its eigenvalues.
2. With A and A_{11} as above, prove that

$$\kappa_2(A_{11}) \leq \kappa_2(A).$$