Precept 7

1. Prove that an (upper) triangular matrix T has its eigenvalues on its diagonal. (Hint: First prove that $\det(T) = \prod T_{ii}$, then use the characteristic polynomial.)

2. Prove that the eigenvalues of the companion matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

are the roots of the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

3. Suppose

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 135 \\ 1 & 0 & 0 & 0 & -297 \\ 0 & 1 & 0 & 0 & 234 \\ 0 & 0 & 1 & 0 & -86 \\ 0 & 0 & 0 & 1 & 15 \end{bmatrix} = QTQ^*$$

where Q is unitary and

$$T = \begin{bmatrix} 3 & * & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Find the roots of the polynomial:

$$p(z) = -2\left((z-3)^5 - 15(z-3)^4 + 86(z-3)^3 - 234(z-3)^2 + 297(z-3) - 135\right)$$

4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite with eigenvalues $\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{\min}$.

(a) Prove the Rayleigh quotient bounds:

$$\lambda_{\min} \leq \frac{x^T A x}{x^T x} \leq \lambda_{\max} \quad (x \neq 0)$$

Find the equality case in each bounds.

(b) Using the Rayleigh quotient bounds from part (a), prove:

(i) $\sqrt{\lambda_{\min}}\|x\| \leq \|x\|_A \leq \sqrt{\lambda_{\max}}\|x\|$ for all x, where $\|x\|_A = \sqrt{x^T Ax}$.

(ii)

$$\frac{\|Ax\|}{\|Ay\|} \leq \kappa_2(A) \cdot \frac{\|x\|}{\|y\|} \quad (x, y \neq 0).$$

5. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. We seek to solve

$$\min_{u\in\mathbb{R}^n} \|A - uu^T\|_F^2.$$

(a) Show that $\nabla f(u) = 4(\|u\|^2 I - A)u$ where $f(u) = \|A - uu^T\|_F^2$. Use the trace identity $\|X\|_F^2 = \text{tr}(X^T X)$, the cyclic property of the trace: $\text{tr}(AB) = \text{tr}(BA)$ and recall that $\nabla(x^T Ax) = 2Ax$ where A is symmetric.

(b) Using part (1), find the critical points of $f(u)$. Be careful with the eigenvalues of A - consider both positive and non-positive cases.

(c) Find the global minimum among all critical points by comparing objective function values.