Precept 8

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Suppose we run a power iteration starting from $v^{(0)} = \begin{bmatrix} a \\ b \end{bmatrix}$ for $a, b \neq 0$ and $a^2 + b^2 = 1$.

True or False: The sequence $v^{(k)}$ will converge to an eigenvector of A. Justify your answer.

2. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Suppose we run a power iteration starting from $v^{(0)} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ with $a^2 + b^2 = 1$. What is the limit of the sequence $v^{(k)}$?

3. Let $A_0 \in \mathbb{R}^{n\times n}$ be symmetric positive definite (SPD). Define the iteration

Algorithm 1 Iteration on SPD matrix

$A_0 \in \mathbb{R}^{n\times n}$ is SPD
for $k = 1, 2, \ldots$ **do**
  Factor $A_{k-1} = G_k^T G_k$ (Cholesky, G_k upper triangular)
  $A_k \leftarrow G_k G_k^T$
end for

(a) Show the iteration is well defined (all steps exist).

(b) Show that the eigenvalues of A_k are the same as the eigenvalues of A_0

(c) Let $A = Q\Lambda Q^T$ with Q orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$. Show that Λ is a fixed point of the iteration (i.e., if $A_{k-1} = \Lambda$, then $A_k = \Lambda$).

4. Let $A \in \mathbb{A}^{n\times n}$ be symmetric and invertible and $Q_0 \in \mathbb{C}^{n\times p}$ have orthonormal columns. Define the iteration

Algorithm 2 Inverse simultaneous iteration

$Q_0 \in \mathbb{C}^{n\times p}$ has orthonormal columns
for $k = 1, 2, \ldots$ **do**
  Solve $AZ_k = Q_{k-1}$
  Compute thin QR: $Z_k = Q_k R_k$
end for

(a) What do you expect the limit of Q_k to be? What assumptions do you need to make about A?

(b) Give the complexity per iteration in each of the following settings (assume $p \ll n$):

  (i) A general symmetric A matrix.

  (ii) A matrix that has been reduced to tridiagonal form.

5. Let $A \in \mathbb{C}^{n \times n}$ be **normal**, i.e., $AA^* = A^*A$.

(a) Prove: that $\|Ax\|_2 = \|A^*x\|_2$ for all $x \in \mathbb{C}^n$.

(b) Prove: if A is also **upper triangular**, then A must be **diagonal**.
Hint: Consider $\|Ae_n\|_2$ and $\|A^*e_n\|_2$ for the standard basis vector e_n.

(c) Use the Schur decomposition to prove: an $n \times n$ matrix is normal **if and only if** it has n orthonormal eigenvectors.
Hint: If $A = QTQ^*$ is the Schur form (unitary Q, upper triangular T), show that A is normal if and only if T is normal.