Precept 9

Commutativity and Similarity

Problem 1.a Let A and B be two $n \times n$ diagonalizable matrices. Show that if A and B share a basis of eigenvectors, that is, there exists V such that $A = V\Lambda_A V^{-1}$ and $B = V\Lambda_B V^{-1}$ where Λ_A and Λ_B are diagonal matrices, then $AB = BA$.

Problem 1.b Let $q_k(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree k, and consider a **non-diagonalizable** matrix A. Show that $q_k(A)A = Aq_k(A)$.

Problem 1.c Let A be SPD, and define its matrix square root as $A^{1/2} = V\Lambda^{1/2}V^{-1}$ where $\Lambda^{1/2}$ is the diagonal matrix of the square roots of the eigenvalues of A. Show that $A^{1/2}q_k(A) = q_k(A)A^{1/2}$.

Evaluating matrix polynomials

Problem 2.1 Let $q(x) = \sum_{i=0}^{m} a_i x^i$ be a degree m polynomial, and let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Show that $q(A)b$ can be computed in m matrix-vector products with A and $\mathcal{O}(mn)$ operations.

Krylov Subspaces

Problem 3.1 Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and $b \in \mathbb{C}^n$. Define the kth Krylov subspace

$$\mathcal{K}_k(A, b) = \operatorname{span}\{b, Ab, \ldots, A^{k-1}b\}.$$

Suppose that for some $k \geq 1$ the subspaces stagnate:

$$\mathcal{K}_{k-1}(A, b) \subset \mathcal{K}_k(A, b) = \mathcal{K}_{k+1}(A, b).$$

1. Show that $A^k b \in \mathcal{K}_k(A, b)$, and therefore there exist coefficients $\alpha_0, \ldots, \alpha_{k-1}$ such that
$$A^k b = \sum_{j=0}^{k-1} \alpha_j A^j b.$$
2. Show that $\alpha_0 \neq 0$. (Hint: What happens if $\alpha_0 = 0$? Use the fact that A is invertible.)

3. Using part (b), construct a vector $x \in \mathcal{K}_k(A, b)$ such that $Ax = b$. Conclude that the exact solution $x_\star = A^{-1}b$ lies in $\mathcal{K}_k(A, b)$.

Conjugate Gradient Method

Problem 4.1 Suppose $A \in \mathbb{R}^{805\times 805}$ is real symmetric with eigenvalues

$$1.00,\ 1.01,\ 1.02,\ \ldots,\ 8.98,\ 8.99,\ 9.00 \quad \text{and} \quad 10,\ 12,\ 16,\ 24.$$

How many steps of the Conjugate Gradient method are sufficient to guarantee that

$$\frac{\|e_k\|_A}{\|e_0\|_A} \le 10^{-6}\,?$$

(Here $e_k = x_\star - x_k$ and $\|v\|_A := \sqrt{v^\top A v}$.)

You may use the following theorems proved in class:

Theorem 1 (CG Error Bound): For CG on SPD A, the A–norm error satisfies

$$\frac{\|e_k\|_A}{\|e_0\|_A} \le \min_{\substack{p\in\Pi_k \\ p(0)=1}} \max_{\lambda\in\sigma(A)} |p(\lambda)|,$$

where Π_k denotes the set of polynomials of degree at most k.

Theorem 2 (Chebyshev Bound): For the shifted–scaled Chebyshev polynomial

$$q_m(t) = \frac{T_m\left(\frac{2t-(a+b)}{b-a}\right)}{T_m\left(\frac{-(a+b)}{b-a}\right)}, \qquad a \le t \le b,$$

we have

$$\max_{t\in[a,b]} |q_m(t)| \ \le\ 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m, \qquad \kappa = \frac{b}{a}.$$

Matrix Functions

Problem 5.1 Suppose $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues. Let $f(z)$ be a polynomial:

$$f(z) = \sum_{i=0}^{m} a_i z^i$$

for some finite degree $m \ge 0$. Let $Q^*AQ = T$ be the Schur form of A (so Q is unitary and T is upper triangular).

1. Show that $f(A) = Q\,f(T)\,Q^*$. Thus, to compute $f(A)$ it suffices to compute $f(T)$. In the rest of the problem you will derive a simple recurrence for $f(T)$.

2. Show that $(f(T))_{ii} = f(T_{ii})$, so the diagonal of $f(T)$ is obtained by applying f entrywise to the diagonal of T. (Hint: Write $T = \Lambda + N$ where Λ is diagonal and N is strictly upper triangular. When expanding $(\Lambda + N)^k$, determine which terms contribute to the diagonal entries.)

3. Show that $T\,f(T) = f(T)\,T$.